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MATHEMATICS

A Theorem on Eigenvectors in Idempotent Spaces

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This paper suggests an algebraic version of the theorem on the existence of eigenvectors for linear operators in abstract idempotent spaces (Theorem 3). Earlier, the theorem on the existence of eigenvectors was only known for the cases of a free finite-dimensional semimodule [3] and for compact operators in semimodules of real continuous functions [4].

1. We use the terminology from [6–8]. Recall that an *idempotent semigroup* (IS) is an additive semigroup with the commutative addition \oplus such that $x \oplus x = x$ for any element x . An arbitrary IS can be treated as an ordered set with the partial order defined as follows: $x \preceq y$ if and only if $x \oplus y = y$. It is easy to see that this order is well-defined and $x \oplus y = \sup(x, y)$. For an arbitrary subset X of an idempotent semigroup, we put $\oplus X = \sup(X)$ and $\wedge X = \inf(X)$ if the corresponding right-hand sides exist. An IS is called *b-complete* (or *boundedly complete*) if any of its subsets bounded from above (including the empty subset) has the least upper bound. In particular, any *b-complete* IS contains zero (denoted by $\mathbf{0}$), which coincides with $\oplus \emptyset$, where \emptyset is the empty set. A homomorphism of *b-complete* idempotent semigroups is called a *b-homomorphism* if $g(\oplus X) = \oplus g(X)$ for any subset X bounded from above. An *idempotent semiring* (ISR) is an IS endowed with an associative multiplication \odot with an identity element (denoted by $\mathbf{1}$) such that both distributivity laws are valid. An ISR is called a *semifield* if any of its nonzero elements has an inverse. An idempotent semifield is called *b-complete* if it is *b-complete* as an IS. The set of all nonzero elements of a *b-complete* semifield is a complete group (in the terminology of [9]). The converse assertion is also valid: adding $\mathbf{0}$ to any complete ordered group and defining addition as \sup , we obtain a *b-complete* semifield. The semifield obtained by this method from the additive group of all real numbers is denoted by \mathbf{R}_{\max} . The theory of ordered groups [9] implies that any *b-complete* semifield is commutative. In any *b-complete* semifield, the generalized distributive laws

$$a \odot (\oplus X) = \oplus (a \odot X), \quad a \odot (\wedge X) = \wedge (a \odot X)$$

are valid; here a is an element of the field and X is a nonempty bounded subset.

An *idempotent semimodule* over an idempotent semiring K is an idempotent semigroup V endowed with a multiplication \odot by elements of K such that, for any

$a, b \in K$ and $x, y \in V$, the usual laws

$$\begin{aligned} a \odot (b \odot x) &= (a \odot b) \odot x, \\ (a \oplus b) \odot x &= a \odot x \oplus b \odot x, \\ a \odot (x \oplus y) &= a \odot x \oplus a \odot y, \\ \mathbf{0} \odot x &= \mathbf{0} \end{aligned}$$

are valid. An idempotent semimodule over an idempotent semifield is called an *idempotent space*. An idempotent b -complete space V over a b -complete semifield K is called an *idempotent b -space* if, for any nonempty bounded subset $Q \subset K$ and any $x \in V$, the relations

$$(\oplus Q) \odot x = \oplus(Q \odot x), \quad (\wedge Q) \odot x = \wedge(Q \odot x)$$

hold. A homomorphism $g : V \rightarrow W$ of b -spaces is called a *b -homomorphism*, or a *b -linear mapping*, if $g(\oplus X) = \oplus g(X)$ for any bounded subset $X \subset V$. More general definitions (for spaces which may not be b -complete) see in [7]. Homomorphisms taking values in K (treated as a semimodule over itself) are called *linear functionals*. A subset of an idempotent space is called a *subspace* if it is closed with respect to addition and multiplication by coefficients. A subspace in a b -space is called a *b -closed subspace* if it is closed with respect to summation over arbitrary bounded (in V) subsets. This subspace has a natural structure of b -space; it is also a b -subspace in V in the sense of [7].

For an arbitrary set X and an idempotent space V over a semifield K , we use $B(X, V)$ to denote the semimodule of all bounded mappings from X into V with pointwise operations. If V is an idempotent b -space, then $B(X, V)$ is a b -space. A mapping f from a topological space X into an ordered set V is called *upper semicontinuous* if, for any $b \in V$, the set $\{x \in X \mid f(x) \succeq b\}$ is closed in X , see [7]. In the case where V is the set of real numbers, this definition coincides with the usual definition of upper semicontinuity of a real function. The set of all bounded upper semicontinuous mappings from X to V is denoted by $USC(X, V)$. If V is a boundedly complete lattice, then $USC(X, V)$ is also a boundedly complete lattice with respect to the pointwise order. If V is an idempotent b -space, then $USC(X, V)$ is also a b -space with respect to the operations $f \oplus g = \sup(f, g)$ and $(k \odot f)(x) = k \odot f(x)$.

2. In what follows, unless otherwise specified, the symbol K stands for a b -complete idempotent semifield and all idempotent spaces are over K .

A subset M of idempotent b -space V is called *wo-closed* if $\wedge X \in M$ and $\oplus X \in M$ for any linearly ordered subset $X \subset M$ in V . A nondecreasing mapping $f : V \rightarrow W$ of b -spaces is called *wo-continuous* if $f(\oplus X) = \oplus f(X)$ and $f(\wedge X) = \wedge f(X)$ for any bounded linearly ordered subset $X \subset V$. Note that an arbitrary isomorphism of ordered sets is *wo-continuous*. It can be shown that the notions of *wo-closedness* and *wo-continuity* coincide with the closedness and continuity with respect to some $T1$ topology defined in an intrinsic way in terms of the order.

Proposition 1. *Suppose that V is an idempotent b -space and W is a wo-closed subsemigroup of V . Then $\oplus X \in W$ for any subset $X \subset W$ bounded in V . In particular, each wo-closed subspace is a b -closed subspace.*

An element x of an idempotent space V is called *Archimedean* if, for any $y \in V$, there exists a coefficient $\lambda \in K$ such that $\lambda \odot x \succeq y$. For an Archimedean element

$x \in V$, the formula $x^*(y) = \bigwedge \{k \in K \mid k \odot x \succeq y\}$ defines a mapping $x^* : V \rightarrow K$. If V is an idempotent b -space, then x^* is a b -linear functional and $x^*(y) \odot x \succeq y$ for any $y \in V$ [6]. We say that an Archimedean element $x \in V$ is *wo-continuous* if the functional x^* is *wo-continuous*, and that an idempotent b -space V is *Archimedean* if V contains a *wo-continuous* Archimedean element.

Proposition 2. *If X is a compact topological space, then $USC(X, K)$ is an Archimedean space and the function \mathbf{e} identically equal to $\mathbf{1}$ is a *wo-continuous* Archimedean element.*

Note that $\mathbf{e}^*(f) = \sup\{f(x) \mid x \in X\}$.

Theorem 1. *Any *wo-closed* subspace of an Archimedean space is an Archimedean space. Any linearly ordered (with respect to the inclusion) family of nonzero *wo-closed* subspaces of an Archimedean space V has a nonzero intersection.*

3. An Archimedean idempotent space V is called *irreducible* with respect to a set G of arbitrary mappings of V into itself if any nonzero *wo-closed* G -invariant subspace W of V coincides with V .

Theorem 2. *Let V be an Archimedean space and G be an arbitrary set of mappings of V into itself. Then V contains a *wo-closed* G -invariant irreducible Archimedean subspace.*

Let V be a semimodule over K . An *eigenvector* of a mapping $g : V \rightarrow V$ corresponding to an *eigenvalue* $\lambda \in K$ is, by definition, a nonzero vector $x \in V$ such that $g(v) = \lambda \odot x$. A semiring K is called *algebraically closed* [3] if, for any elements $x \in K$ and any positive n , there exists an element $y \in K$ such that $y^n = x$. For example, \mathbf{R}_{\max} is a b -complete algebraically closed semifield.

Theorem 3. *An arbitrary b -linear mapping of an Archimedean idempotent space over an algebraically closed semifield into itself has an eigenvector.*

Proposition 3. *Suppose that V is an Archimedean semimodule over \mathbf{R}_{\max} , g is a homomorphism (or even an arbitrary homogeneous mapping) of the semimodule V into itself, and $x, y \in V$ are eigenvectors of g corresponding to eigenvalues $p, q \in \mathbf{R}_{\max}$, respectively. Then (i) $x \preceq y$ implies $p \preceq q$ and (ii) all eigenvalues corresponding to Archimedean eigenvectors coincide.*

4. Let V be a b -space. A subset $W \subset V$ is called a \wedge -subspace if it is closed with respect to multiplication by scalars and taking greatest lower bounds of nonempty subsets. By this definition, any such W is a boundedly complete lattice with respect to other inherited from the ambient space. Therefore, any \wedge -subspace $W \subset V$ can be treated as a semimodule with respect to the inherited multiplication by scalars and operations $x \oplus_W y = \sup(x, y)$, where \sup is over W . In what follows, all \wedge -subspaces are considered as semimodules with respect to these operations. The definitions immediately imply that any \wedge -subspace of a b -space is a b -space. It is easy to show that $USC(X, V)$ is a \wedge -subspace in $B(X, V)$ for any b -space V and any topological space X .

Theorem 4. *If V is an Archimedean b -space and $x \in V$ is a *wo-continuous* Archimedean element, then any \wedge -subspace W of V containing x is an Archimedean b -space.*

Theorem 5. *An idempotent b -space V over an algebraically closed b -complete semifield K is Archimedean if and only if there exists a space of the form $USC(X, K)$, where X is a compact topological space, such that V is isomorphic to its \wedge -subspace containing constants.*

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